Ex Suppose a cylindrical water tank of height 1 m with cross-sectional area 1 m² is being filled at a rate of 1 l/s with water. Water leaks out a small hole in the bottom. When the tank is completely full, the leak rate is 2 l/s. Assume the draining tank obeys Torricelli's Law.

a) Can the tank ever empty?
b) Can the tank ever overflow?
c) Is there a stable, equilibrium water depth?

d) Write the DE for h(t), the water depth.
Derive the general equation for \( h(t) \) in terms of the tank area \( A \), maximum depth \( D \), inflow rate \( Q_i \), outflow rate at maximum depth \( Q_o \). Scale the resulting equation to be as simple as possible.
5 Minute Mathematics Break.

A computational method is used to determine a drag coefficient for flow over a proposed aircraft wing design. Computations are first done on a design for which the drag is known exactly. The following errors are seen as the grid spacing $h$ is refined:

<table>
<thead>
<tr>
<th>$h$</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.04002</td>
</tr>
<tr>
<td>0.05</td>
<td>0.00881</td>
</tr>
<tr>
<td>0.025</td>
<td>0.00201</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.00049</td>
</tr>
</tbody>
</table>

What is the order of convergence of the method?
Ex. Approximate the solution of

\[ y' = \sqrt{y^2 + t^2} \]

with \( y(0) = 1 \) at \( T = 0.2 \) using 2 steps of FE with step size \( h = 0.1 \).

Ex. Consider the DE system

\[
\begin{align*}
\frac{dx}{dt} &= x + y + t \\
\frac{dy}{dt} &= x - y
\end{align*}
\]

with initial data \( x(0) = 1, \ y(0) = 0 \).

Use Euler's method with 2 steps of size \( h = 0.1 \) to approximate the solution at \( t = 0.2 \) (both \( x(0.2) \) and \( y(0.2) \))
5 Minute Mathematics Break.

Consider the following DE for \( x(t) \):

\[
\dot{x} = ax - bx^2 + cx^3. \quad (a, b, c \text{ given positive})
\]

Scale \( x \) and \( t \) to make the first two terms on the RHS of the equation as simple as possible.

You should have a single parameter left in the equation, a combination of \( a, b, \) and \( c \).
5 Minute Mathematics Break.

Convert the following 2nd order DE for \( x(t) \) to a first order system:

\[ \ddot{x} + t^2 \dot{x} + x^2 = 0. \]

If you have time, apply FE stepping to this problem with

\[ x(0) = 1, \quad \dot{x}(0) = 1 \]

Do 2 steps of size \( h = 0.1 \).
Ex. Consider the problem for $y(t)$:
\[ \frac{dy}{dt} = t + y^2 \quad \text{with} \quad y(0) = 1. \]
Approximate $y(0.1)$ using one step of BE time stepping.

Ex. Use one step of Improved Euler to approximate $y(0.2)$ where $y(t)$ solves
\[ y' = 3 - t + y \]
with $y(0) = 1$. 